

The ground state of long-range Schrödinger equations and static $q\bar{q}$ potential

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ABSTRACT: Motivated by the recent results in [arXiv:1601.05679](https://arxiv.org/abs/1601.05679) about the quark-antiquark potential in $\mathcal{N} = 4$ SYM, we reconsider the problem of computing the asymptotic weak-coupling expansion of the ground state energy of a certain class of 1d Schrödinger operators $-\frac{d^2}{dx^2} + \lambda V(x)$ with long-range potential $V(x)$. In particular, we consider even potentials obeying $\int_{\mathbb{R}} dx V(x) < 0$ with large x asymptotics $V \sim -a/x^2 - b/x^3 + \dots$. The associated Schrödinger operator is known to admit a bound state for $\lambda \rightarrow 0^+$, but the binding energy is rigorously non-analytic at $\lambda = 0$. Its asymptotic expansion starts at order $\mathcal{O}(\lambda)$, but contains higher corrections $\lambda^n \log^m \lambda$ with all $0 \leq m \leq n-1$ and standard Rayleigh-Schrödinger perturbation theory fails order by order in λ . We discuss various analytical tools to tame this problem and provide the general expansion of the binding energy at $\mathcal{O}(\lambda^3)$ in terms of quadratures. The method is tested on a soluble potential that is fully under control, and on various non-soluble cases as well. A supersymmetric case, arising in the study of the quark-antiquark potential in $\mathcal{N} = 6$ ABJ(M) theory, is also exploited to provide a further non-trivial consistency check. Our analytical results confirm at third order a remarkable exponentiation of the leading infrared logarithms, first noticed in $\mathcal{N} = 4$ SYM where it may be proved by Renormalization Group arguments. We prove this interesting feature at all orders at the level of the Schrödinger equation for general potentials in the considered class.

KEYWORDS: AdS-CFT Correspondence, Supersymmetric gauge theory

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1 Introduction

Let us consider the Schrödinger operator in one dimension

$$H = -\frac{d^2}{dx^2} + \lambda V(x), \quad x \in \mathbb{R}, \quad \lambda > 0. \quad (1.1)$$

A variational argument shows that if $V(x) \leq 0$, with $V(x) < 0$ on an open set, and $V(x) \rightarrow 0$ at infinity, then H has a bound state for all $\lambda > 0$. The lowest eigenvalue $E_0(\lambda)$ is real analytic for $\lambda > 0$ under mild conditions on V [1]. Interesting questions are analyticity at $\lambda = 0$ and the existence of bound states for small λ when V is somewhere positive.¹ The analysis of [5] showed that when $\int_{\mathbb{R}} dx V(x) \leq 0$ and $\int_{\mathbb{R}} dx (1+x^2) |V(x)| < \infty$, there is a unique bound state for small λ . The ground state (binding) energy may be estimated in this case by the formula

$$\sqrt{-E_0(\lambda)} = -\frac{1}{2} \lambda \int_{\mathbb{R}} dx V(x) - \frac{1}{4} \lambda^2 \int_{\mathbb{R}} dx dy |x-y| V(x) V(y) + \mathcal{O}(\lambda^2). \quad (1.2)$$

If for some $a > 0$ we have $\int_{\mathbb{R}} dx e^{a|x|} V(x) < \infty$, then the quantity $\sqrt{-E_0(\lambda)}$ is analytic at $\lambda = 0$.² If the potential is such that $\int_{\mathbb{R}} dx (1+x^2) |V(x)| = \infty$, and $V(x) \sim -a|x|^{-\beta}$ at infinity, the results of [11] imply the following. For $2 < \beta < 3$, the estimate (1.2) is still valid. For $\beta = 2$, there is a unique bound state for $\lambda \rightarrow 0^+$ provided that $\int_{\mathbb{R}} dx V(x) \leq 0$. However, (1.2) is violated because the r.h.s. develops a term $\lambda^2 \log \lambda$. If $1 < \beta < 2$, there are infinitely many bound states for any $\lambda > 0$ and (1.2) is valid when restricted to the first term only, i.e.

$$\sqrt{-E_0(\lambda)} = -\frac{1}{2} \lambda \int_{\mathbb{R}} dx V(x) + \mathcal{O}(\lambda). \quad (1.3)$$

Here, we shall be concerned with the $\beta = 2$ case. Further discussion of other long-range cases may be found in [12, 13] and in the review [14]. Thus, in this paper, we shall address the problem (1.1) with a potential in the class defined by the conditions

$$V(x) \stackrel{x \rightarrow \infty}{\sim} -a x^{-2} - b |x|^{-3} + \dots, \quad \int_{\mathbb{R}} dx V(x) < 0, \quad (1.4)$$

and, for simplicity, we shall also assume it to be even $V(x) = V(-x)$. The modified form of (1.2) under the conditions in (1.4) has been found in [11] and reads

$$\sqrt{-E_0(\lambda)} = -\left[\frac{1}{2} \lambda + a \lambda^2 \log \lambda \right] \int_{\mathbb{R}} dx V(x) + \mathcal{O}(\lambda^2). \quad (1.5)$$

A non-analytic logarithmic factor enhances the second order term. Standard Rayleigh-Schrödinger perturbation theory completely fails and this term is infinite.³ In general, it is

¹The large λ limit is completely different, see for instance [2, 3], although in some cases, scale invariance connects it to the weak coupling regime [4].

²This short-range case is much simpler [6] and may be treated by standard perturbation theory. For instance, a quite compact extension of (1.2) to order $\mathcal{O}(\lambda^4)$ is discussed in [7–9] and may be easily extended at higher orders. The first term of (1.2) may be found immediately by applying the Feynman-Hellman theorem [10].

³It is important to emphasize that we are not dealing with the divergence of the perturbative series due to a vanishing radius of convergence. The long-distance behaviour of the potential is such that the single terms in the Rayleigh-Schrödinger perturbative expansion are separately divergent.

very difficult to recover results like (1.5) from some kind of regularized perturbation theory. A very illustrative example are crude infrared cutoffs like $|x| \leq L$.⁴ Finite size effects may be suppressed at large L but by factors like $e^{-\lambda g}$ or similar. Expanding first in λ shows that the cutoff dependence is increasingly bad at higher orders.

A modern application of results like (1.5) is to the study of the quark-antiquark static potential in flat space for $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N_c)$.⁵ The static potential is extracted from a pair of anti-parallel Wilson lines separated by the distance r [18, 19]. In the planar limit $N_c \rightarrow \infty$, with fixed 't Hooft coupling $\hat{\lambda} = g_{\text{YM}}^2 N_c$,⁶ we can obtain the quark-antiquark potential $-\frac{1}{r} \Omega(\hat{\lambda})$ from the expectation value of the associated Wilson loop. The function $\Omega(\hat{\lambda})$ is known at 3 loops at weak coupling [18, 20–25] and at one loop at strong coupling [26–29]. This construction can be extended by introducing an angular parameter ϑ that takes into account the relative flavours of the quark and antiquark. The generalized potential $\Omega(\hat{\lambda}, \vartheta)$ has been recently studied in [30] by means of the Quantum Spectral Curve methods developed in [31]. One of the remarkable results of [30] is the analytic weak coupling expansion of $\Omega(\hat{\lambda}, \vartheta)$ up to 7 loops, i.e. at order $\mathcal{O}(\lambda^7)$. A particularly interesting limit is the double scaling regime $\vartheta \rightarrow i\infty$ with fixed $\hat{\lambda} e^{-i\vartheta}$. This limit resums the ladder diagrams in the gauge theory and the quark-antiquark potential is obtained as the ground state energy of a 1d Schrödinger equation with potential in the class (1.4). In the notation of (1.1), the problem addressed in [30] corresponds to the specific potential

$$V(x) \equiv V_I(x) = -\frac{1}{1+x^2}, \quad (1.6)$$

with Schrödinger coupling $\lambda = \frac{\hat{\lambda} e^{-i\vartheta}}{16\pi^2}$ and identification $E_0 = -\frac{1}{4} \Omega^2(\lambda)$. The first three terms of the expansion of $\sqrt{-E_0}$ may be written [30]

$$\begin{aligned} \sqrt{-E_0(\lambda)} &= \frac{\pi}{2} \lambda + \pi (\log \lambda + 2\mathcal{L} - 1) \lambda^2 \\ &+ \pi \left[\log^2 \lambda + (4\mathcal{L} + 1) \log \lambda - \frac{1}{12} (-48\mathcal{L}^2 - 24\mathcal{L} + \pi^2 + 42) \right] \lambda^3 + \dots, \end{aligned} \quad (1.7)$$

where $\mathcal{L} = \log \sqrt{2e^{\gamma_E}} \pi$, and γ_E denotes the Euler-Mascheroni constant. Inspection of the $\mathcal{O}(\lambda^7)$ additional terms in (1.7) shows that each power λ^n comes together with all logarithms $\log^m \lambda$ with $0 \leq m \leq n-1$. Also, the leading and next-to-leading logarithms seem to be captured by the resummation Ansatz

$$\sqrt{-E_0(\lambda)} \Big|_{\text{NLO}} = \frac{\pi}{2} \lambda e^{2\lambda \mathcal{L}} (1 + 6\lambda^2 \mathcal{L}), \quad \mathcal{L} = \log \lambda + 2\mathcal{L} - 1, \quad (1.8)$$

⁴The study of the infrared problems that appear when $V(x)$ is treated as a perturbation of the laplacian dates back to the initial developments of quantum mechanics. A celebrated example is E. P. Wigner's discussion of the hydrogen atom in the classical paper [15], see also [16].

⁵Another (older) interesting physical application of (1.1) with potential in the class (1.4) occurs in the study of wetting transition of 2d surfaces [17]. In that context, the interface free-energy density is given in the thermodynamic limit by the lowest eigenvalue E_0 of (1.1) where x denotes the perpendicular distance of the interface from the substrate, and $V(x)$ is the potential well that localizes the interface below the transition temperature, related to λ .

⁶We introduce a slightly unconventional hat in the notation for the 't Hooft coupling to avoid confusion with the coupling λ in (1.1) that we prefer to keep in order to match the conventions in the mathematical physics literature.

whose leading part is a simple exponentiation. The expansion (1.7) and the resummation (1.8) pose several interesting questions. The first is whether the structure of (1.7) is general, i.e. if it is true that — at least at third order — for all potentials in the class (1.4) one has

$$\sqrt{-E_0(\lambda)} = c_{1,0} \lambda + (c_{2,1} \log \lambda + c_{2,0}) \lambda^2 + (c_{3,2} \log^2 \lambda + c_{3,1} \log \lambda + c_{3,0}) \lambda^3 + \dots \quad (1.9)$$

The second question concerns the general validity of resummation formulas like (1.8). In the context of the static quark-antiquark potential, resummation of leading and next-to-leading logarithms has been performed by a Renormalization Group (RG) analysis in [20, 23]. In particular, leading logarithms simply exponentiate and, since they are computed exactly by the ladder approximation, this explain the first term in (1.8). Nevertheless, it would be interesting to understand whether such exponentiation is expected for a general potential and can be determined at the level of the Schrödinger equation. This question is particularly intriguing because in gauge theories the RG exponentiation — at leading order — encodes the factorization scale dependence that remains after cancellation between real and virtual infrared divergences. In the Schrödinger equation, this deep machinery is somewhat absent and exponentiation of logarithms of λ is less clear and more elusive.

The plan and content of this paper is the following. In section (2), we discuss a fully solvable potential in the class (1.4) where we provide the expansion of the binding energy at high order showing that it admits the structure in (1.9) as well as an exponentiation formula similar to (1.8). In section (3), we present a first simple extension of the result (1.5), by computing the subleading coefficient $c_{2,0}$ in (1.9). The derivation is rather straightforward, but cannot be easily extended at higher orders. To overcome these difficulties, we apply in section (4) the general approach of matched asymptotic expansions to our problem. As a preparation, section (4.1) is devoted to discuss a toy-model that captures the essence of the method. Then, in section (4.2), we apply it to the determination of (1.9) and we compute the complete third order expansion for a general potential. Our main result, summarized in (4.39) and (4.40), is tested on various solvable and non-solvable examples in section (5). In particular, section (5.5) presents several high precision numerical tests of the third order expansions to appreciate its accuracy at small coupling, despite being only asymptotic. As a final test, we discuss in section (6) a consistency check of our expansion, when applied to a supersymmetric potential arising in the study of the quark-antiquark potential in $\mathcal{N} = 6$ ABJ(M) theory. Finally, our third order expansion suggests exponentiation of the leading logarithms in the form (see the first term of (1.8) as a special case)

$$\sqrt{-E_0(\lambda)} = -\frac{1}{2} \lambda e^{2a\lambda \log \lambda} \int_{\mathbb{R}} dx V(x) + \text{subleading logarithms}. \quad (1.10)$$

In section (7), we prove that this result holds at all orders. Finally, section (8) summarises our conclusions and outlines possible further directions.

2 A solvable example

The potential (1.6) may be treated by the Quantum Spectral Curve as a special limit. To improve our knowledge, we also analyze the following solvable case discussed in [11, 32]

$$V(x) \equiv V_{\text{II}}(x) = -\frac{1}{4} \frac{1}{(1+|x|)^2}. \quad (2.1)$$

Its ground state energy $\alpha \equiv \sqrt{-E_0(\lambda)}$ obeys the exact equation

$$\alpha \frac{\partial}{\partial \alpha} K_\nu(\alpha) + \frac{1}{2} K_\nu(\alpha) = 0, \quad \nu = \frac{1}{2} \sqrt{1-\lambda}, \quad (2.2)$$

where $K_\nu(\alpha)$ are modified Bessel functions of the second kind. Expanding (2.2) at small λ , we get the asymptotic series

$$\begin{aligned} \sqrt{-E_0(\lambda)} = & \frac{1}{4} \lambda + \frac{L}{8} \lambda^2 + \left(\frac{L^2}{32} + \frac{5L}{32} - \frac{11}{128} \right) \lambda^3 \\ & + \left(\frac{L^3}{192} + \frac{7L^2}{64} + \frac{21L}{256} + \frac{1}{768} (14\zeta_3 - 71) \right) \lambda^4 \\ & + \left(\frac{L^4}{1536} + \frac{35L^3}{768} + \frac{175L^2}{1024} + \frac{L(28\zeta_3 - 89)}{3072} + \frac{2352\zeta_3 - 5833}{73728} \right) \lambda^5 \\ & + \left(\frac{L^5}{15360} + \frac{23L^4}{1536} + \frac{859L^3}{6144} + \frac{L^2(7\zeta_3 + 444)}{3072} \right. \\ & \left. + \frac{L(5376\zeta_3 - 19105)}{147456} + \frac{5355\zeta_3 + 558\zeta_5 - 9176}{184320} \right) \lambda^6 + \dots, \end{aligned} \quad (2.3)$$

where $L = \log \lambda - \log 2 + \gamma_E + \frac{1}{2}$. All the leading $\lambda^{n+1} L^n$ and subleading $\lambda^{n+2} L^n$ logarithms are captured by the following formula (see appendix (A) for a proof)

$$\sqrt{-E_0(\lambda)} \Big|_{\text{NLO}} = \frac{1}{4} \lambda \left[e^{\frac{1}{2} \lambda L} \left(1 - \frac{1}{4} \lambda + \frac{3}{8} \lambda^2 L \right) + \frac{1}{4} \lambda e^{\frac{3}{2} \lambda L} \right]. \quad (2.4)$$

The first term in the square bracket is similar to the resummation (1.8). Some differences in the other terms are not surprising because here the $\mathcal{O}(x^{-3})$ subleading term at infinity is non zero, and may be important. Collecting data from (1.7) and (2.3), we obtain the reference table table 1 for the coefficients appearing in (1.9) up to third order: this is what we want to be able to compute for a general potential in the class (1.4). Of course, the (leading) second order rigorous result (1.5) is fully consistent with the entries in table 1.

3 The complete second order expansion

A first simple extension of the result (1.5) obtained in [11] concerns the determination of the coefficient $c_{2,0}$ in (1.9). The analysis in [11] is based on the results of [5] that imply that the ground state energy $E_0 = -\alpha^2$ is obtained from the condition

$$\det(1 + \lambda K_\alpha) = 0, \quad (3.1)$$

	$c_{1,0}$	$c_{2,1}$	$c_{2,0}$	$c_{3,2}$	$c_{3,1}$	$c_{3,0}$
$V_I(x)$	$\frac{\pi}{2}$	π	$\pi(2\mathcal{L} - 1)$	π	$\pi(4\mathcal{L} + 1)$	$-\frac{1}{12}(-48\mathcal{L}^2 - 24\mathcal{L} + \pi^2 + 42)$
$V_{II}(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}(\gamma_E - \log 2 + \frac{1}{2})$	$\frac{1}{32}$	$\frac{1}{16}(3 + \gamma_E - \log 2)$	$\frac{1}{32}(\gamma_E - \log 2)(\gamma_E - \log 2 + 6)$

Table 1. Summary table showing the coefficient of the asymptotic expansion of the binding energy at third order for the test case potentials $V_I(x)$ and $V_{II}(x)$.

where K_α is the Birman-Schwinger operator [1]

$$K_\alpha(x, y) = \frac{1}{2\alpha} |V(x)|^{1/2} e^{-\alpha|x-y|} |V(y)|^{1/2} \text{sign}(V(y)). \quad (3.2)$$

We can expand (3.1) as follows

$$\begin{aligned} \det(1 + \lambda K_\alpha) &= e^{\text{tr} \log(1 + \lambda K_\alpha)} = 1 + \lambda \text{tr} K_\alpha + \frac{\lambda^2}{2} \left[(\text{tr} K_\alpha)^2 - \text{tr} K_\alpha^2 \right] \\ &\quad + \frac{\lambda^3}{6} \left[(\text{tr} K_\alpha)^3 - 3 \text{tr} K_\alpha \text{tr} K_\alpha^2 + 2 \text{tr} K_\alpha^3 \right] + \dots \end{aligned} \quad (3.3)$$

At second order, (3.3) may be written in the simple form⁷

$$0 = 1 + \frac{\lambda}{2\alpha} \int_{\mathbb{R}} dx V(x) + \frac{\lambda^2}{8\alpha^2} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy V(x) V(y) (1 - e^{-2\alpha|x-y|}) + \dots \quad (3.4)$$

The rigorous analysis of [11] leads to the result (1.5) for the λ and $\lambda^2 \log \lambda$ term. If we want to capture the λ^2 term (without logarithmic enhancement), we need to study in details the small α limit of the second order correction

$$F(\alpha) = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy V(x) V(y) (1 - e^{-2\alpha|x-y|}). \quad (3.5)$$

To this aim, we define the Fourier transform of $V(x)$ with the convention

$$\tilde{V}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dx V(x) e^{i\omega x}, \quad (3.6)$$

and easily obtain the alternative form of $F(\alpha)$

$$F(\alpha) = 2\pi \int_{\mathbb{R}} d\omega |\tilde{V}(\omega)|^2 \left[\delta(\omega) - \frac{2\alpha}{\pi(4\alpha^2 + \omega^2)} \right] = 2 \int_{\mathbb{R}} d\omega \frac{|\tilde{V}(0)|^2 - |\tilde{V}(2\alpha\omega)|^2}{1 + \omega^2}. \quad (3.7)$$

For a specific potential, one can study this integral for $\alpha \rightarrow 0$,⁸ but here we want to discuss (3.7) in general terms. One straightforward approach is the following. Since $V(x)$

⁷An alternative expression for the second order term is

$$\int_{-\infty}^{\infty} dx dy V(x) V(y) (1 - e^{-2\alpha|x-y|}) = 2 \int_{-\infty}^{\infty} dx \int_0^{\infty} dz V(x) V(x+z) (1 - e^{-2\alpha z}).$$

⁸For instance, if the Mellin transform of $|\tilde{V}(\omega)|^2$ is known, then it is easy to obtain the small α asymptotic expansion of $F(\alpha)$.

is real, $\tilde{V}(\omega)^* = \tilde{V}(-\omega)$ and $U(\omega) \equiv |\tilde{V}(\omega)|^2$ is even. The expression (3.7) can be split as follows

$$\begin{aligned} F(\alpha) &= 4 \int_0^\infty d\omega \frac{U(0) - U(2\alpha\omega)}{1 + \omega^2} = 4 \left(\int_0^{1/(2\alpha)} + \int_{1/(2\alpha)}^\infty \right) d\omega \frac{U(0) - U(2\alpha\omega)}{1 + \omega^2} \\ &= 8\alpha \left(\int_0^1 + \int_1^\infty \right) d\omega \frac{U(0) - U(\omega)}{4\alpha^2 + \omega^2} \end{aligned} \quad (3.8)$$

The first integral is computed by adding and subtracting in the numerator $\omega U'(0^+)$ and splitting the elementary part that can be computed explicitly. In what remains, as well as in the second integral, we can safely send $\alpha \rightarrow 0$ if we are interested in the $\mathcal{O}(\alpha \log \alpha)$ and $\mathcal{O}(\alpha)$ terms of $F(\alpha)$. In conclusion,

$$\begin{aligned} F(\alpha) &= 8\alpha \left[U'(0^+) (\log \alpha + \log 2) \right. \\ &\quad \left. + \int_0^1 \frac{U(0) + U'(0^+)\omega - U(\omega)}{\omega^2} + \int_1^\infty \frac{U(0) - U(\omega)}{\omega^2} \right] + \mathcal{O}(\alpha^2). \end{aligned} \quad (3.9)$$

Given

$$F(\alpha) = \alpha (F_1 \log \alpha + F_0) + \mathcal{O}(\alpha^2), \quad (3.10)$$

we obtain from (3.4)

$$0 = 1 + \frac{\lambda}{2\alpha} \int_{\mathbb{R}} dx V + \frac{\lambda^2}{8\alpha} (F_1 \log \alpha + F_0) + \dots \quad (3.11)$$

Denoting for brevity

$$\bar{V} = \int_{\mathbb{R}} dx V(x), \quad (3.12)$$

and inverting (3.11), we find

$$\alpha = -\frac{\bar{V}}{2} \lambda - \frac{\lambda^2}{8} \left[F_1 \log \lambda + F_0 + F_1 \log \left(-\frac{1}{2} \bar{V} \right) \right] + \dots \quad (3.13)$$

In conclusion, we find the second order terms in the expansion (1.9) with the following coefficients

$$\begin{aligned} c_{1,0} &= -\frac{1}{2} \bar{V}, \\ c_{2,1} &= -U'(0^+), \\ c_{2,2} &= -U'(0^+) \log(-\bar{V}) - \int_0^1 \frac{U(0) + U'(0^+)\omega - U(\omega)}{\omega^2} - \int_1^\infty \frac{U(0) - U(\omega)}{\omega^2}. \end{aligned} \quad (3.14)$$

Notice that, to match (3.3), we need to show that $U'(0^+) = a \bar{V}$. This is slightly non trivial. For an even potential $\tilde{V}(\omega)$ is also even, and then $U'(0^+) = 2 \tilde{V}(0) \tilde{V}'(0^+)$. Then,

$$\tilde{V}'(\omega) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x V(x) e^{i\omega x} = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x \left[V(x) + \frac{a}{1+x^2} \right] e^{i\omega x} - \frac{i a}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{x}{1+x^2} e^{i\omega x}. \quad (3.15)$$

In the first integral we can safely send $\omega \rightarrow 0$ and then it vanishes because it is odd under $x \rightarrow -x$. The remaining piece gives

$$\tilde{V}'(0^+) = \frac{a}{\sqrt{2\pi}} \lim_{\omega \rightarrow 0^+} \int_{\mathbb{R}} \frac{x \sin(\omega x)}{1+x^2} = \lim_{\omega \rightarrow 0^+} \frac{a}{\sqrt{2\pi}} \frac{\pi \omega e^{-|\omega|}}{|\omega|} = a \sqrt{\frac{\pi}{2}}. \quad (3.16)$$

Finally, using $\tilde{V}(0) = \frac{1}{\sqrt{2\pi}} \bar{V}$, we match (3.3).

3.1 Check with the potential $V_I(x) = -\frac{1}{1+x^2}$

In this case,

$$\bar{V} = -\pi, \quad U(\omega) = \frac{\pi}{2} e^{-2\omega}, \quad U'(0^+) = -\pi. \quad (3.17)$$

Also, we can compute⁹

$$\begin{aligned} \int_0^1 d\omega \frac{U(0) + U'(0^+) \omega - U(\omega)}{\omega^2} &= \frac{1}{2} \pi \left(2 \operatorname{Ei}(-2) + 1 + \frac{1}{e^2} - 2\gamma_E - 2 \log 2 \right), \\ \int_1^\infty d\omega \frac{U(0) - U(\omega)}{\omega^2} &= \frac{1}{2} \pi \left(-2 \operatorname{Ei}(-2) + 1 - \frac{1}{e^2} \right). \end{aligned} \quad (3.18)$$

Replacing in (3.14), we obtain

$$c_{1,0} = \frac{\pi}{2}, \quad c_{2,1} = \pi, \quad c_{2,2} = \pi (-1 + \gamma_E + \log 2 + \log \pi) = \pi (2\mathcal{L} - 1), \quad (3.19)$$

in full agreement with the entries in table 1.

3.2 Check with the potential $V_{II}(x) = -\frac{1}{4(1+|x|)^2}$

In this case,

$$\tilde{V}(\omega) = -\frac{2 - \omega(2\operatorname{Ci}(\omega) \sin \omega + (\pi - 2\operatorname{Si}(\omega)) \cos \omega)}{4\sqrt{2\pi}} \quad (3.20)$$

and using $\tilde{V}(\omega) = -\frac{1}{2\sqrt{2\pi}} + \frac{1}{4}\sqrt{\frac{\pi}{2}}\omega + \dots$, we find

$$\bar{V} = -\frac{1}{2}, \quad U(0) = \frac{1}{8\pi}, \quad U'(0^+) = -\frac{1}{8}. \quad (3.21)$$

Also, we can compute the relevant non trivial integrals numerically at high precision finding

$$\begin{aligned} I_1 &= \int_0^1 d\omega \frac{U(0) + U'(0^+) \omega - U(\omega)}{\omega^2} = -0.172399851825784556337918255\dots, \\ I_2 &= \int_1^\infty d\omega \frac{U(0) - U(\omega)}{\omega^2} = 0.03774789371309294876210\dots, \end{aligned} \quad (3.22)$$

Remarkably, we checked that up to 100 digits we have

$$I_1 + I_2 = -\frac{1}{16}(1 + 2\gamma_E). \quad (3.23)$$

⁹Here, the exponential integral function is

$$\operatorname{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt.$$

Replacing in (3.14), we obtain

$$c_{1,0} = -\frac{1}{4}, \quad c_{2,1} = -\frac{1}{8}, \quad c_{2,2} = \frac{1}{16}(1 + 2\gamma_E - 2\log 2). \quad (3.24)$$

again in full agreement with table 1.

3.3 Leading logarithms at third order

One may attempt to apply this approach to work out the third order contribution to $\sqrt{-E_0}$, but this is definitely a non-trivial task, although the leading order term $\sim \lambda^3 \log^2 \lambda$ is heuristically derived in appendix (B). In the next section, we shall introduce an alternative method that is powerful enough to fully compute the third order expansion and, in principle, also higher order terms.

4 Matched Asymptotic Expansion, the complete third order

A completely different approach is based on the method of matched asymptotic expansions for boundary layer problems [33, 34]. Although the idea is the same as in the standard examples of singular perturbation of differential equations [35], the specific application to the present setup is rather different and somewhat unusual.¹⁰ Earlier studies in the context of the quark-antiquark potential may be found in [37, 38]. For the potential $V_I(x)$, this method has been successfully exploited in [21]. The method is fully general and we shall present its application for a generic potential in the class (1.4). A recent application to a different kind of Schrödinger problem may be found in [39].¹¹ We now first present a simple example illustrating the procedure and then apply the method to our case.

4.1 A toy model

Let us consider the first order boundary problem

$$\begin{aligned} \psi'(x) &= \left(\alpha + \frac{\lambda}{1+x} \right) \psi(x), \\ \psi(0) &= 1, \quad \psi(\lambda^{-1}) = 1. \end{aligned} \quad (4.1)$$

Although apparently contrived, this problem captures the essential features of our problem. We are interested in the $\lambda \rightarrow 0^+$ limit and there are two boundary conditions, one in the UV region, at small x , and one in the IR region, at large x . The two boundary conditions determine α as a function of λ . Indeed, the exact solution of (4.1) before imposing the boundary condition at $x = \lambda^{-1}$ is

$$\psi(x) = e^{\alpha x} (1+x)^\lambda. \quad (4.2)$$

¹⁰The application of matched asymptotic expansion methods to the Schrödinger equation in the semi-classical limit is of course a text-book topic, see for instance [36] for a clean pedagogical presentation. In that context, matching is applied around the inversion points.

¹¹The main new complication to be discussed here is the presence of non-analytic terms in a boundary problem eigenvalue. This is a feature that, for instance, is absent in the analysis of the Mathieu equation in [35].

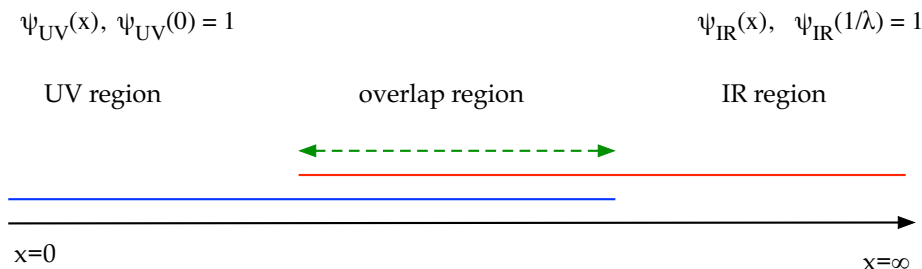


Figure 1. Qualitative picture illustrating the Matched Asymptotic Expansion procedure.

Imposing $\psi(\lambda^{-1}) = 1$, we obtain the following asymptotic expansion¹²

$$\alpha = -\lambda^2 \log \left(1 + \frac{1}{\lambda} \right) = \lambda^2 \log \lambda - \lambda^3 + \frac{1}{2} \lambda^4 + \dots, \quad (4.3)$$

that is the result we want to derive by an alternative route. To this aim, we apply the ideas of matched asymptotic expansions and look for two approximate solutions of (4.1). The first is $\psi_{\text{IR}}(x)$ and is valid around $x = \infty$, with $\psi_{\text{IR}}(\lambda^{-1}) = 1$. The second is $\psi_{\text{UV}}(x)$ and must be used around $x = 0$, with $\psi_{\text{UV}}(0) = 1$. The two solutions must be compared by imposing the symbolic matching

$$\lim_{x \rightarrow \infty} \psi_{\text{UV}}(x) = \lim_{x \rightarrow 0} \psi_{\text{IR}}(x). \quad (4.4)$$

For this procedure to make sense, the precise meaning of the relation (4.4) is that we need $\psi_{\text{UV}}(x) \simeq \psi_{\text{IR}}(x)$ in an intermediate overlap region, as shown in figure 1. Besides, in our application the matching will be imposed perturbatively in λ .

Following this approach, the first step is to consider the approximate equation that is obtained at large x and solve it with the IR boundary condition:

$$\begin{aligned} \psi'_{\text{IR}}(x) &= \left[\alpha + \lambda \left(\frac{1}{x} - \frac{1}{x^2} + \frac{\xi_3}{x^3} - \frac{\xi_4}{x^4} \right) \right] \psi_{\text{IR}}(x), \\ \psi_{\text{IR}}(\lambda^{-1}) &= 1. \end{aligned} \quad (4.5)$$

We have kept four terms in the large x expansion of the potential. This is instructive with the aim of keeping track of the role of the various corrections. In other words, the coefficients ξ_3, ξ_4 are useful to identify terms that depend on including ($\xi = 1$) or excluding ($\xi = 0$) a certain subleading part of V . The solution of (4.5) is

$$\psi_{\text{IR}}(x) = \lambda^\lambda x^\lambda \exp \left[-\frac{\alpha}{\lambda} + \alpha x + \lambda \left(\frac{1}{x} - \frac{\xi_3}{2x^2} + \frac{\xi_4}{3x^3} \right) - \lambda^2 + \frac{\lambda^3}{2} \xi_3 - \frac{\lambda^4}{3} \xi_4 \right]. \quad (4.6)$$

Now, we make the educated guess¹³

$$\alpha = A \lambda^2 \log \lambda + B \lambda^3 + \dots, \quad (4.7)$$

¹²We remark that in this simple example, the non-analyticity due to logarithms is limited to the first term, while the remainder is actually a power series with unit convergence radius.

¹³The appearance of a double expansion in λ and $\log \lambda$ is familiar in the context of matched asymptotic techniques where logarithmic terms are commonly named as *switchback* corrections [40].

and expand (4.6) at small λ , and fixed x . This is an expansion in powers of λ with possible powers of $\log \lambda$. Explicitly, we find

$$\psi_{\text{IR}}(x) = 1 + \lambda \psi_{\text{IR},1}(x) + \lambda^2 \psi_{\text{IR},2}(x) + \cdots, \quad (4.8)$$

with

$$\begin{aligned} \psi_{\text{IR},1}(x) &= (1-A) \log \lambda + \log x + \frac{1}{x} - \frac{\xi_3}{2x^2} + \frac{\xi_4}{3x^3}, \\ \psi_{\text{IR},2}(x) &= Ax \log \lambda + \frac{1}{2}[(A-1)^2 \log^2 \lambda - 2(A-1) \log \lambda \log x - 2(B+1) + \log^2 x] \\ &\quad + \frac{\log x + (1-A) \log \lambda}{x} + \frac{1 - \xi_3 [\log x + (1-A) \log \lambda]}{2x^2} \\ &\quad + \frac{2\xi_4 [\log x + (1-A) \log \lambda] - 3\xi_3}{6x^3} + \frac{\frac{\xi_3^2}{8} + \frac{\xi_4}{3}}{x^4} - \frac{\xi_3 \xi_4}{6x^5} + \frac{\xi_4^2}{18x^6}. \end{aligned} \quad (4.9)$$

In the UV region, we consider the unmodified initial problem, but impose only the boundary condition at $x = 0$

$$\begin{aligned} \psi'_{\text{UV}}(x) &= \left(\alpha + \frac{\lambda x}{1+x} \right) \psi_{\text{UV}}(x), \\ \psi_{\text{UV}}(0) &= 1. \end{aligned} \quad (4.10)$$

To match (4.8), we solve (4.10) perturbatively in λ , taking into account (4.7),

$$\psi_{\text{UV}}(x) = 1 + \lambda \psi_{\text{UV},1}(x) + \lambda^2 \psi_{\text{UV},2}(x) + \cdots, \quad \psi_{\text{UV},n}(0) = 0. \quad (4.11)$$

This gives immediately

$$\begin{aligned} \psi_{\text{UV},1}(x) &= \log(x+1), \\ \psi_{\text{UV},2}(x) &= Ax \log \lambda + \frac{1}{2} \log^2(x+1). \end{aligned} \quad (4.12)$$

Finally, we attempt to match the two solutions (4.8) and (4.11) by evaluating the ratio

$$R(x, \lambda) = \frac{\psi_{\text{IR}}(x)}{\psi_{\text{UV}}(x)} = 1 + \lambda R_1(x, \lambda) + \lambda^2 R_2(x, \lambda) + \cdots. \quad (4.13)$$

The contribution from the UV region must be expanded at large x where the overlap region is located. Doing so, we obtain at first order

$$R_1(x, \lambda) = (1-A) \log \lambda + \frac{1 - \xi_3}{2x^2} + \frac{\xi_4 - 1}{3x^3} + \frac{1}{4x^4} - \frac{1}{5x^5} + \cdots. \quad (4.14)$$

Thus, we see that matching ($R_n = 0$) requires $A = 1$ in agreement with (4.3). This does not depend on the ξ -terms, i.e. the x^{-3} and x^{-4} corrections to V included in (4.5). The correct choice $\xi_3 = \xi_4 = 1$ cancels the various corrections in (4.14), but we repeat that this is irrelevant to fix A . With the choice $A = 1$, we can inspect $R_2(x, \lambda)$ and find

$$R_2(x, \lambda) = -B - 1 + \frac{(\xi_3 - 1)^2}{8x^4} - \frac{(\xi_3 - 1)(\xi_4 - 1)}{6x^5} + \frac{4\xi_4^2 - 8\xi_4 - 9\xi_3 + 13}{72x^6} + \frac{6\xi_3 + 5\xi_4 - 11}{60x^7} + \cdots. \quad (4.15)$$

Again, the coefficient B is fixed at $B = -1$ in agreement with (4.3). The first two subleading corrections in (4.15) vanish if we include the x^{-3} correction in (4.5). Doing so, i.e. taking $\xi_3 = 1$, we get

$$R_2(x, \lambda)|_{B=-1, \xi_3=1} = \frac{(\xi_4 - 1)^2}{18x^6} + \frac{\xi_4 - 1}{12x^7} + \frac{47 - 32\xi_4}{480x^8} + \frac{10\xi_4 - 19}{180x^9} + \dots, \quad (4.16)$$

showing that the inclusion of the x^{-4} correction in (4.5), i.e. taking $\xi_4 = 1$, cancel two additional subleading terms.

4.2 Application to the third order correction

We now apply to our main problem the procedure that worked for the toy model (4.1).

4.2.1 Analysis of the infrared region

Let us consider an even potential admitting the expansion for $x \rightarrow +\infty$

$$V(x) = -\frac{a}{x^2} - \frac{b}{x^3} + \mathcal{O}(x^{-4}). \quad (4.17)$$

The large x equation

$$-\psi''(x) + \left(\alpha^2 - \frac{a\lambda}{x^2}\right)\psi(x) = 0, \quad (4.18)$$

has the following solution decaying exponentially at infinity

$$\psi(x) = \psi^{(1)}(x) = \sqrt{\frac{2\alpha x}{\pi}} K_{\frac{1}{2}\sqrt{1-4a\lambda}}(\alpha x). \quad (4.19)$$

The second independent solution is

$$\psi^{(2)}(x) = \sqrt{\frac{2\alpha x}{\pi}} I_{\frac{1}{2}\sqrt{1-4a\lambda}}(\alpha x). \quad (4.20)$$

Taking into account the next to leading correction to the potential at large x , the IR solution is

$$\psi_{\text{IR}}(x) = \psi^{(1)}(x) + \frac{\pi\lambda}{2\alpha} \left[b\psi^{(1)}(x) \int^x dx' \frac{\psi^{(2)}(x')\psi^{(1)}(x')}{(x')^3} - b\psi^{(2)}(x) \int^x dx' \frac{\psi^{(1)}(x')^2}{(x')^3} \right] + \dots \quad (4.21)$$

As we explained, we want to expand this solution in the limit $\alpha x \rightarrow 0$, and then $\lambda \rightarrow 0$. It is convenient to set $\alpha = A(\lambda)\lambda$ and we find (c_n are constant terms dependent of A, a, b, λ , but independent of x)

$$\begin{aligned} \psi_{\text{IR}}(x) = & 1 + \lambda \left[-\frac{b}{2x} + a \log x - Ax + c_1 \right] + \\ & + \lambda^2 \left[\frac{A^2 x^2}{2} + x \left(\frac{A(2a^2 L_A - 4a^2 - Ab)}{2a} + aA \log x \right) \right. \\ & \left. + \frac{1}{2} \log x (2a^2 L_A + 2a^2 - Ab) + \frac{1}{2} a^2 \log^2 x - \frac{ab L_A + 1 + \log x}{2} + c_2 \right] + \mathcal{O}(\lambda^3). \end{aligned} \quad (4.22)$$

where $L_A = \log(2A\lambda) + \gamma_E$. The higher orders λ^n are complicated but have a similar structure.

4.2.2 Analysis of the ultraviolet region

In the UV region, we first expand the exact equation

$$-\psi''(x) + (\alpha^2 + \lambda V(x))\psi(x) = 0, \quad (4.23)$$

in powers of λ

$$\psi(x) = 1 + \lambda \psi_1(x) + \lambda^2 \psi_2(x) + \lambda^3 \psi_3(x) + \dots \quad (4.24)$$

This leads to

$$\begin{aligned} \psi_1''(x) &= V(x), \\ \psi_2''(x) &= A^2 + V(x) \psi_1(x), \\ \psi_3''(x) &= A^2 \psi_1(x) + V(x) \psi_2(x), \quad \text{and so on.} \end{aligned} \quad (4.25)$$

Now, we want to solve these equations with the condition $\psi'_n(0) = 0$. The values $\psi_n(0)$ are not relevant because they may be absorbed by a redefinition of $\psi(x)$ by a factor that depends on λ but not on x . We shall choose $\psi_n(0) = 0$. Thus, (4.25) has to be solved with

$$\psi_n(0) = \psi'_n(0) = 0. \quad (4.26)$$

Given the solution, we want to expand it at large x and compare with (4.22), after the educated guess, including switchback logarithmic terms,

$$A(\lambda) = c_{1,0} + (c_{2,1} \log \lambda + c_{2,0}) \lambda + (c_{3,2} \log^2 \lambda + c_{3,1} \log \lambda + c_{3,0}) \lambda^3 + \dots \quad (4.27)$$

To see how this works, let us begin with $\psi_1(x)$, i.e.¹⁴

$$\psi_1(x) = \int_0^x dx' (x - x') V(x'). \quad (4.28)$$

The large x expansion is obtained as follows

$$\begin{aligned} \psi_1(x) &= \int_0^x dx' (x - x') \left[V(x') + \frac{a}{(x'+1)^2} \right] - \int_0^x dx' \frac{a(x-x')}{(x'+1)^2} \\ &= \int_0^\infty dx' (x - x') \left[V(x') + \frac{a}{(x'+1)^2} \right] - \int_x^\infty dx' (x - x') \left[V(x') + \frac{a}{(x'+1)^2} \right] \\ &\quad - a(x - \log(x+1)). \end{aligned} \quad (4.29)$$

Computing the first integral and expanding inside the second one, we obtain

$$\psi_1(x) = C_1 + \frac{1}{2} \bar{V} x + a \log x - \frac{b}{2x} + \dots, \quad (4.30)$$

with

$$C_1 = - \int_0^\infty dx x \left[V(x) + \frac{a}{(x+1)^2} \right]. \quad (4.31)$$

¹⁴The double integration of (4.25) is trivially reduced to (4.28). Alternatively, two derivatives of (4.28) give the first of (4.25).

Comparing the UV expansion (4.30) with the IR expansion (4.22), we see that we indeed recover the correct value of $c_{1,0}$ in agreement with the known result. At the next order, i.e. solving for the function ψ_2 , we find

$$\begin{aligned}\psi_2(x) &= \frac{A^2 x^2}{2} + \int_0^x dx' (x - x') V(x') \psi_1(x') \\ &= \frac{A^2 x^2}{2} + x \int_0^x dx' \left[V(x') \psi_1(x') + \frac{a\bar{V}}{2} \frac{1}{x' + 1} \right] - x \int_0^x dx' \left[\frac{a\bar{V}}{2} \frac{1}{x' + 1} \right] \\ &\quad - \int_0^x dx' x' \left[V(x') \psi_1(x') + \frac{a\bar{V}}{2} \frac{1}{x' + 1} + \frac{aC_1 + \frac{\bar{V}}{2}(a+b) + a^2 \log x'}{(x' + 1)^2} \right] \\ &\quad + \int_0^x dx' x' \left[\frac{a\bar{V}}{2} \frac{1}{x' + 1} + \frac{aC_1 + \frac{\bar{V}}{2}(a+b) + a^2 \log x'}{(x' + 1)^2} \right]\end{aligned}\tag{4.32}$$

Doing the previous split in the integrals that cannot be computed by elementary means, we obtain

$$\begin{aligned}\psi_2(x) &= C_2 + \frac{A^2}{2} x^2 + x \left(k - \frac{a\bar{V}}{2} \log x + \frac{a\bar{V}}{2} \right) + \frac{a^2}{2} \log^2 x \\ &\quad + \left(a^2 + aC_1 + \frac{b\bar{V}}{2} \right) \log x - \frac{ab(1 + \log x) + bC_1}{2x} + \frac{b^2}{12x^2} + \dots,\end{aligned}\tag{4.33}$$

where¹⁵

$$\begin{aligned}k &= \int_0^\infty dx \left[V(x) \psi_1(x) + \frac{a\bar{V}}{2} \frac{1}{1+x} \right], \\ C_2 &= - \int_0^\infty dx x \left[V(x) \psi_1(x) + \frac{a\bar{V}}{2} \frac{1}{x+1} + \frac{aC_1 + \frac{\bar{V}}{2}(a+b) + a^2 \log x}{(x+1)^2} \right] - \frac{a^2}{6} (\pi^2 - 6) - \frac{a\bar{V}}{2}.\end{aligned}\tag{4.34}$$

Finally, let us consider ψ_3 , we need the expansion of it up to a constant because this is the last stage of our computation.

$$\begin{aligned}\psi'_3(x) &= \int_0^x dx' \left[A^2 \psi_1(x') + V(x') \psi_2(x') \right] \\ &= A^2 \int_0^x dx' \left[\psi_1(x') - C_1 - \frac{\bar{V}}{2} x' - a \log x' + \frac{b}{2(x' + 1)} \right] \\ &\quad + A^2 \int_0^x dx' \left[C_1 + \frac{\bar{V}}{2} x' + a \log x' - \frac{b}{2(x' + 1)} \right] \\ &\quad + \int_0^x dx' \left[V(x') \psi_2(x') + \frac{aA^2}{2} + \frac{-a^2\bar{V} \log x' + a^2\bar{V} + 2ak + A^2b}{2(x' + 1)} \right] \\ &\quad - \int_0^x dx' \left[\frac{aA^2}{2} + \frac{-a^2\bar{V} \log x' + a^2\bar{V} + 2ak + A^2b}{2(x' + 1)} \right].\end{aligned}\tag{4.35}$$

¹⁵An alternative more explicit form is $k = \frac{\bar{V}}{2} (a - C_1) - 2 \int_0^\infty dx \int_0^\infty dy x V(x) V(x+y)$.

Integrating,

$$\begin{aligned}\psi_3(x) = & C_3 + \frac{A^2 \bar{V}}{12} x^3 + \frac{A^2}{2} (a \log x - 2a + C_1) x^2 \\ & + x \left(\frac{a^2 \bar{V}}{4} \log^2 x + \left(1 - \frac{\pi^2}{12} \right) a^2 \bar{V} - \log x (a(a\bar{V} + k) + A^2 b) + ak + A^2(b + k') + k'' \right) \\ & + \mathcal{O}(x^0 \log^n x),\end{aligned}\tag{4.36}$$

where

$$\begin{aligned}k' = & \int_0^\infty dx \left[\psi_1(x) - C_1 - \frac{\bar{V}}{2} x - a \log x + \frac{b}{2(x+1)} \right], \\ k'' = & \int_0^\infty dx \left[V(x) \psi_2(x) + \frac{aA^2}{2} + \frac{-a^2 \bar{V} \log x + a^2 \bar{V} + 2ak + A^2 b}{2(x+1)} \right],\end{aligned}\tag{4.37}$$

are new constants to be determined by quadrature.

4.2.3 Matching UV and IR, final result

The coefficients of the expansion (3.1) may be found by comparing the expansions in the IR and UV regions. After straightforward calculations, the general result reads

$$\begin{aligned}c_{1,0} = & -\frac{1}{2} \bar{V}, \\ c_{2,1} = & -a \bar{V}, \\ c_{2,0} = & \frac{1}{2} \bar{V} (-2a \log(-\bar{V}) + a(1 - 2\gamma_E) + C_1) - k, \\ c_{3,2} = & -a^2 \bar{V}, \\ c_{3,1} = & -\frac{b}{2} \bar{V}^2 + a \bar{V} (-2a \log(-\bar{V}) - 2a(1 + \gamma_E) + C_1) + 2ak, \\ c_{3,0} = & \frac{1}{16} \bar{V}^2 (-8b \log(-\bar{V}) + (5 - 8\gamma_E)b - 4k') \\ & + \bar{V} \left\{ \frac{1}{12} \left[(24 - 24\gamma_E - 12\gamma_E^2 + \pi^2) a^2 + 6a(2\gamma_E + 1)C_1 - 6C_1^2 + 6C_2 \right] \right. \\ & \quad \left. - a^2 \log^2(-\bar{V}) + a(C_1 - 2(1 + \gamma_E)a) \log(-\bar{V}) \right\} \\ & - 2ak \log(-\bar{V}) - a(2\gamma_E + 1)k + C_1 k - k''\end{aligned}\tag{4.38}$$

It is convenient to write the asymptotic expansion (3.1) by isolating a relevant IR scale and making explicit the simplest coefficients

$$\begin{aligned}\sqrt{-E_0(\lambda)} = & -\frac{1}{2} \bar{V} \lambda \left[1 + 2a \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) \lambda \right. \\ & \left. + \left(2a^2 \log^2 \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) + (6a^2 + b\bar{V}) \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) + \tilde{c}_{3,0} \right) \lambda^2 + \dots \right].\end{aligned}\tag{4.39}$$

with

$$\begin{aligned}\log \lambda_{\text{IR}} &= \frac{C_1}{2a} - \frac{k}{a\bar{V}} - \log(-\bar{V}) - \gamma_E + \frac{1}{2}, \\ \tilde{c}_{3,0} &= -\frac{2k^2}{\bar{V}^2} + \frac{2k'' - 2ak}{\bar{V}} + \frac{-(9 + \pi^2)a^3 + 6a^2C_1 + 3a(C_1^2 - 2C_2) - 6bk}{6a} \\ &\quad + \frac{1}{8}\bar{V} \left[b \left(\frac{4C_1}{a} - 1 \right) + 4k' \right].\end{aligned}\quad (4.40)$$

Notice that the exponentiation of the leading logs is recovered at this order. The quantities in (4.40) can be computed in terms of (3.12), (4.31), (4.34), and (4.37). These require quadratures that involves the functions $\psi_1(x)$ and $\psi_2(x)$ whose expression must be computed from, see (4.28),

$$\psi_1(x) = \int_0^x dx' (x - x') V(x'), \quad \psi_2(x) = \int_0^x dx' (x - x') \left[A^2 + V(x') \psi_1(x') \right], \quad (4.41)$$

and the quantity A may be replaced by $-\frac{1}{2}\bar{V}$ at this order.

5 Checks and applications

5.1 Check with the potential $V_I(x) = -\frac{1}{1+x^2}$

For the potential V_I , we obtain the following explicit functions in (4.41) ($\psi_2(x)$ is real for $x > 0$ although this is not obvious from the following expression)

$$\begin{aligned}\psi_1(x) &= \frac{1}{2} \log(x^2 + 1) - x \arctan x, \\ \psi_2(x) &= \frac{1}{24} (24ix \text{Li}_2(-e^{2i \arctan x}) + 3 \log(x^2 + 1)(\log(x^2 + 1) + 4) \\ &\quad - 12x \left(\log(x^2 + 1) + 4 \log\left(\frac{2i}{x+i}\right) + 2 \right) \arctan x + \pi^2 x (3x + 2i) \\ &\quad + 12(1 + 2ix) \arctan^2 x).\end{aligned}\quad (5.1)$$

The various constants needed in (4.40) can be computed in closed form and read

$$\begin{aligned}\bar{V} &= -\pi, \quad a = 1, \quad b = 0, \quad C_1 = 1, \quad k = -\pi \log 2, \quad C_2 = 3 + \frac{\pi^2}{8}, \\ k' &= \frac{\pi}{4}, \quad k'' = -\frac{\pi^3}{8} - \pi \log^2 2 - \pi \log 2.\end{aligned}\quad (5.2)$$

Replacing them in (4.38), we match the content of table 1. In the form (4.39), it reads

$$\begin{aligned}\sqrt{-E_0(\lambda)} &= \frac{\pi}{2} \lambda \left[1 + 2 \log\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) \lambda + \left(2 \log^2\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) + 6 \log\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) - 4(2\mathcal{L}^2 + \mathcal{L} - 1) \right. \right. \\ &\quad \left. \left. + \frac{8\mathcal{L}^2 + 4\mathcal{L} - 7}{\pi} - \frac{\pi}{6} \right) \lambda^2 + \dots \right].\end{aligned}\quad (5.3)$$

with $\log \lambda_{\text{IR}} = 1 - 2\mathcal{L}$.

5.2 Check with the potential $V_{\text{II}}(x) = -\frac{1}{4(1+|x|)^2}$

For the potential V_{II} , we obtain

$$\begin{aligned}\psi_1(x) &= \frac{1}{4}(\log(x+1) - x), \\ \psi_2(x) &= \frac{1}{32}((x-6)x + \log(x+1)(2x + \log(x+1) + 6)).\end{aligned}\tag{5.4}$$

The various constants are now

$$\begin{aligned}\bar{V} &= -\frac{1}{2}, & a &= \frac{1}{4}, & b &= -\frac{1}{2}, & C_1 &= 0, & k &= -\frac{1}{8}, & C_2 &= \frac{1}{16}, \\ k' &= \frac{1}{4}, & k'' &= -\frac{13}{128} - \frac{\pi^2}{384}.\end{aligned}\tag{5.5}$$

Again, replacing them in (4.38), we match the content of table 1. In the form (4.39), it reads

$$\sqrt{-E_0(\lambda)} = \frac{1}{4}\lambda \left[1 + \frac{1}{2} \log\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) \lambda + \left(\frac{1}{8} \log^2\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) + \frac{5}{8} \log\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) - \frac{11}{32} \right) \lambda^2 + \dots \right]. \tag{5.6}$$

with $\log \lambda_{\text{IR}} = \log 2 - \gamma_E - \frac{1}{2}$.

5.3 Expansion for new non-soluble potentials, examples

As an application of our method to a new potentials that are not soluble, nor already treated by other methods, we consider the following two cases

$$V_{\text{III}}(x) = -\frac{|x|}{4(1+|x|)^3}, \quad V_{\text{IV}}(x) = -\frac{x^2}{(1+x^2)^2}.\tag{5.7}$$

For the potential V_{III} , we obtain

$$\begin{aligned}\psi_1(x) &= \frac{1}{4} \log(x+1) - \frac{x(x+2)}{8(x+1)}, \\ \psi_2(x) &= \frac{1}{384(x+1)^2} \left(3x^4 - 38x^3 + 12x^3 \log(x+1) - 99x^2 + 12x^2 \log^2(x+1) \right. \\ &\quad + 72x^2 \log(x+1) - 60x + 24x \log^2(x+1) + 12 \log^2(x+1) \\ &\quad \left. + 120x \log(x+1) + 60 \log(x+1) \right).\end{aligned}\tag{5.8}$$

The various constants are

$$\begin{aligned}\bar{V} &= -\frac{1}{4}, & a &= \frac{1}{4}, & b &= -\frac{3}{4}, & C_1 &= -\frac{1}{8}, & k &= -\frac{1}{12}, & C_2 &= -\frac{1}{192}, \\ k' &= \frac{1}{4}, & k'' &= -\frac{319}{4608} - \frac{\pi^2}{768}.\end{aligned}\tag{5.9}$$

Replacing them in (4.38), we obtain the new asymptotic expansion

$$\sqrt{-E_0(\lambda)} = \frac{1}{8}\lambda \left[1 + \frac{1}{2} \log\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) \lambda + \left(\frac{1}{8} \log^2\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) + \frac{9}{16} \log\left(\frac{\lambda}{\lambda_{\text{IR}}}\right) - \frac{43}{144} \right) \lambda^2 + \dots \right]. \tag{5.10}$$

with $\log \lambda_{\text{IR}} = 2 \log 2 - \gamma_E - \frac{13}{12}$. A similar calculation may be performed for the potential V_{IV} . We omit the details and just write down the final expansion

$$\sqrt{-E_0(\lambda)} = \frac{\pi}{4} \lambda \left[1 + 2 \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) \lambda + \left(2 \log^2 \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) + 6 \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) - \frac{17\pi^2}{48} - \frac{121}{32} - 2 \log^2 2 - \frac{7}{2} \log 2 \right) \lambda^2 + \dots \right]. \quad (5.11)$$

with $\log \lambda_{\text{IR}} = -\gamma_E + \frac{3}{8} - \log \pi$.

5.4 Simple extensions

We remark that our expressions are sufficiently simple to deal with potentials that depend on parameters. One neat example is the potential

$$V(x) = -\frac{\beta - 1 + |x|}{4(1 + |x|)^3}, \quad \beta \geq 0. \quad (5.12)$$

For this potential we obtain the following generalization of the $\beta = 1$ case in (5.10),

$$\sqrt{-E_0(\lambda)} = \frac{\beta}{8} \lambda \left[1 + \frac{1}{2} \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) \lambda + \left(\frac{1}{8} \log^2 \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) + \frac{6 + 4\beta - \beta^2}{16} \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) - \frac{1}{72\beta^2} + \frac{5}{144\beta} - \frac{13}{144} - \frac{217\beta}{576} + \frac{25\beta^2}{144} - \frac{5\beta^3}{192} \right) \lambda^2 + \dots \right]. \quad (5.13)$$

with $\log \lambda_{\text{IR}} = 2 \log 2 - \gamma_E - \log \beta - \frac{1}{3\beta} - \frac{7}{6} + \frac{5}{12} \beta$. In general, it is clear that if the expansion may be worked out in details for V_1 and V_2 , then it may be computed for any linear combination of them. Also, the class of potentials may be enlarged. One interesting example is that of a potential with asymptotic form at infinity including a term $\sim \log |x|/x^4$. Just to give a simple example, we studied the case

$$V(x) = -\frac{1}{(1 + |x|)^2} + \frac{9}{2} \frac{\log(1 + |x|)}{(1 + |x|)^4}, \quad (5.14)$$

where the relative weight of the first and second terms has been chosen to have $\bar{V} = -1$. We find in this case,

$$\sqrt{-E_0(\lambda)} = \frac{1}{2} \lambda \left[1 + 2 \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) \lambda + \left(2 \log^2 \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) + 8 \log \left(\frac{\lambda}{\lambda_{\text{IR}}} \right) - \frac{141990237413}{2^6 3^2 5^6 7^4} \right) \lambda^2 + \dots \right], \quad (5.15)$$

with $\log \lambda_{\text{IR}} = -\gamma_E - \frac{3313}{3000}$.

5.5 Numerical tests

We present a numerical test of the accuracy of the asymptotic expansions (5.3), (5.6), and the new (5.10), (5.11). The convergence radius of these expansions is expected to be zero, but looking at sufficiently small λ , it is possible to appreciate the improvement associated with the higher order terms. The following tables will present the first, second, and third order expansion of $\sqrt{-E_0(\lambda)}$ for the four considered potentials. The last column, labeled

exact is obtained by finding the ground state in the interval $x \in [0, L]$.¹⁶ For sufficiently large L , the number of stable digits in the result doubles as L is doubled showing exponential convergence at large L . Our results at $\lambda = 10^{-1}, 10^{-2}, 10^{-3}$ (and $5 \cdot 10^{-4}$ for V_{IV}) are

$$V_I(x) = -\frac{1}{1+x^2},$$

	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$
1 st	0.1570796327	0.0157079633	0.0015707963
2 st	0.1291982378	0.0147057709	0.0015535406
3 st	0.1076056359	0.0146919732	0.0015535679
exact	0.1169558258	0.0147016491	0.0015535766

(5.16)

$$V_{II}(x) = -\frac{1}{4(1+|x|)^2},$$

	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$
1 st	0.025	0.0025	0.00025
2 st	0.0226018542	0.0024472362	0.0002491845
3 st	0.0223311706	0.0024470475	0.0002491848
exact	0.0223479210	0.0024470589	0.0002491848

(5.17)

$$V_{III}(x) = -\frac{|x|}{4(1+|x|)^3},$$

	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$
1 st	0.0125	0.00125	0.000125
2 st	0.0112322935	0.0012229318	0.0001245854
3 st	0.0111166333	0.0012228830	0.0001245856
exact	0.0111254679	0.0012228880	0.0001245856

(5.18)

$$V_{IV}(x) = -\frac{x^2}{(1+x^2)^2},$$

	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$	$\lambda = 5 \cdot 10^{-4}$
1 st	0.0785398163	0.0078539816	0.0007853982	0.0003926991
2 st	0.0635286655	0.0073421809	0.0007766633	0.0003902432
3 st	0.0520846293	0.0073351273	0.0007766773	0.0003902461
exact	0.0578934148	0.0073412803	0.0007766830	0.0003902468

(5.19)

From the above tables, it is possible to check that the third order gives a sensible improvement at these small values of λ . The pattern is the same for all the considered potentials and, in particular, confirms the new expansion (5.10). Just as a final example,

¹⁶Of course, for V_{II} we solve (2.2).

we also report the numerical results associated with the potential (5.14)

$$V(x) = -\frac{1}{(1+|x|)^2} + \frac{9}{2} \frac{\log(1+|x|)}{(1+|x|)^4},$$

	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$
1 st	0.05	0.005	0.0005
2 st	0.0437896391	0.0047076379	0.0004947738
3 st	0.0384057384	0.0047012055	0.0004947769
exact	0.0384464983	0.0047018683	0.0004947770

(5.20)

6 A further consistency check from $\mathcal{N} = 6$ ABJ(M) theory

Apart from general questions, our initial motivation came from the Quantum Spectral Curve results on the static quark-antiquark potential in $\mathcal{N} = 4$ super Yang-Mills gauge theory. Quite recently, new interesting results appeared in the study of the cusp anomalous dimension in $\mathcal{N} = 6$ ABJ(M) theory. A scaling limit has been identified where ladder diagrams dominate and the Bethe-Salpeter equation leads again to a Schrödinger problem [41]. This problem is studied for generic Wilson loop Euclidean cusp angle φ . The anti-parallel limit $\varphi \rightarrow \pi$ is the one relevant for the calculation of the static potential, but requires some care as it is singular as in $\mathcal{N} = 4$ SYM. On general grounds, it is expected that the larger amount of supersymmetry should be responsible for cancellation of the non-analytic logarithms [42]. Technically, in the ladder approximation, this feature is due to the fact that the Schrödinger potential in the ladder approximation is supersymmetric, i.e. it has the structure of Witten's supersymmetric quantum mechanics [43]. In particular, the $\mathcal{N} = 6$ ABJ(M) version of the potential (1.6) — relevant for $\mathcal{N} = 4$ SYM — reads¹⁷

$$H = -\frac{d^2}{dx^2} - \frac{\lambda^2}{x^2 + 1} - \frac{\lambda}{(x^2 + 1)^{3/2}}, \quad \lambda > 0. \quad (6.1)$$

The exact bound state wave function is $\psi(x) = \mathcal{N} \exp(-\lambda \sqrt{x^2 + 1})$, with binding energy $E_0 = -\lambda^2$, without (logarithmic) corrections. We can put this problem in the form (1.1), by considering the potential

$$V(x) = -\frac{\mu}{x^2 + 1} - \frac{1}{(x^2 + 1)^{3/2}}, \quad (6.2)$$

where μ has to be replaced by λ at the end of the calculation. This is a well-defined procedure and the replacement $\mu \rightarrow \lambda$ simply induces a rearrangement of the asymptotic expansion. Applying (4.39) to (6.2), we obtain the non-trivial expansion at generic fixed μ

$$\begin{aligned} \sqrt{-E_0(\lambda; \mu)} = \frac{2 + \pi \mu}{2} \lambda & \left[1 + 2\mu \log\left(\frac{\lambda}{\lambda_{\text{IR}}(\mu)}\right) \lambda + \left(2\mu^2 \log^2\left(\frac{\lambda}{\lambda_{\text{IR}}(\mu)}\right) \right. \right. \\ & \left. \left. - (2 + \pi \mu - 6\mu^2) \log\left(\frac{\lambda}{\lambda_{\text{IR}}(\mu)}\right) - \frac{\pi}{2\mu} - \frac{\pi^2}{8} + \mathcal{O}(\mu)\right) \lambda^2 \right] + \mathcal{O}(\lambda^4), \end{aligned} \quad (6.3)$$

¹⁷We thanks the authors of [41, 42], and in particular D. Seminara, for providing the details of the $\varphi \rightarrow \pi$ limit in $\mathcal{N} = 6$ ABJ(M) theory.

where

$$\log \lambda_{\text{IR}}(\mu) = \frac{\pi + 4\mu + 2\pi\mu^2}{4\mu + 2\pi\mu^2} + \frac{2 - \pi\mu}{2 + \pi\mu} \log 2 - \log(2 + \pi\mu) - \gamma_E. \quad (6.4)$$

Replacing $\mu = \lambda$, the terms in (6.3) and (6.4) rearrange nicely to give

$$\sqrt{-E_0(\lambda; \lambda)} = \lambda + \mathcal{O}(\lambda^4), \quad (6.5)$$

in agreement with the exact binding energy $E_0 = -\lambda^2$. This is a further non-trivial check of our expansion (4.39).

7 Universal resummation of leading infrared logarithms

The resummation results in (1.8) and (2.4) are consistent with (4.39), (4.40). It is tempting to conjecture that leading logarithms indeed exponentiate at all orders, i.e. that the expression

$$\sqrt{-E_0(\lambda)} \Big|_{\text{LO}} = -\frac{\bar{V}}{2} \lambda e^{2a\lambda \log \lambda}, \quad (7.1)$$

captures all terms of the form $\lambda^{n+1} \log^n \lambda$. Exponentiation results like (7.1) appears in applications of matched asymptotic expansions often together with Renormalization Group techniques [44–47]. However, a general treatment is missing and we devote this section to the analysis of (7.1) in our specific setup. A physical argument for (7.1) is that the non-analytic logarithms of the coupling are an infrared effect related to the long-distance behaviour of the potential. This somewhat explains why resummation involves essentially the coefficient a in (1.4). Nevertheless, it is amusing that some global information about the potential is needed in the form of the prefactor in (7.1) that depends on \bar{V} . An interesting further check of (7.1) may be presented by considering a soluble cut-off version of the potential $V_{\text{II}}(x)$, as discussed in appendix (C).

A general proof of (7.1) may be given exploiting the clever splitting of the Birman-Schwinger operator (3.2) introduced in [12]. To this aim, we define the operators P_α , Q_α with integral kernels

$$\begin{aligned} P_\alpha(x, y) &= -\frac{1}{2} |V(x)|^{1/2} e^{-\alpha|x|} e^{-\alpha|y|} |V(y)|^{1/2} \text{sign}(V(x)), \\ Q_\alpha(x, y) &= -\frac{1}{2} |V(x)|^{1/2} \left[e^{-\alpha|x-y|} - e^{-\alpha|x|} e^{-\alpha|y|} \right] |V(y)|^{1/2} \text{sign}(V(x)). \end{aligned} \quad (7.2)$$

According to the Birman-Schwinger principle [1], the Schrödinger operator in (1.1) has the eigenvalue $-\alpha^2$ if and only if $P_\alpha + Q_\alpha$ has eigenvalue α/λ or, equivalently, if $1 - \lambda\alpha^{-1}(P_\alpha + Q_\alpha)$ is not invertible — compare with (3.1). Since P_α is rank 1, this translates into the condition

$$\begin{aligned} \alpha &= \lambda \text{Tr} \left(P_\alpha (1 - \lambda\alpha^{-1} Q_\alpha)^{-1} \right) \\ &= \lambda \text{Tr}(P_\alpha) + \frac{\lambda^2}{\alpha} \text{Tr}(P_\alpha Q_\alpha) + \frac{\lambda^3}{\alpha^2} \text{Tr}(P_\alpha Q_\alpha^2) + \dots \end{aligned} \quad (7.3)$$

The first term is

$$\text{Tr}(P_\alpha) = -\frac{1}{2} \int_{\mathbb{R}} dx V(x) e^{-2\alpha|x|} = -\frac{1}{2} \bar{V} + \int_0^\infty dx V(x) (1 - e^{-2\alpha x}). \quad (7.4)$$

The second term can be computed by adding and subtracting the leading large distance term $-a/(1+x)^2$ to $V(x)$ and we obtain

$$\text{Tr}(P_\alpha) = -\frac{1}{2} \bar{V} + 2a\alpha \log \alpha + \text{subleading terms}. \quad (7.5)$$

Let us analyze the next terms

$$\begin{aligned} \text{Tr}(P_\alpha Q_\alpha^{n-1}) &= \frac{(-1)^n}{2^n} \int_{\mathbb{R}} dx_1 \cdots dx_n V(x_1) \cdots V(x_n) e^{-\alpha|x_1|-\alpha|x_2|} \\ &\quad \left[e^{-\alpha|x_2-x_3|} - e^{-\alpha|x_2|} e^{-\alpha|x_3|} \right] \cdots \left[e^{-\alpha|x_n-x_1|} - e^{-\alpha|x_n|} e^{-\alpha|x_1|} \right] \\ &= \frac{(-1)^n}{2^{n-1}} \int_{x_i \geq 0} dx_1 \cdots dx_n V(x_1) \cdots V(x_n) e^{-\alpha x_1 - \alpha x_2} \\ &\quad \left[e^{-\alpha|x_2-x_3|} + e^{-\alpha(x_2+x_3)} - 2e^{-\alpha x_2} e^{-\alpha x_3} \right] \cdots \left[e^{-\alpha|x_n-x_1|} + e^{-\alpha(x_n+x_1)} - 2e^{-\alpha x_n} e^{-\alpha x_1} \right] \\ &= \frac{(-1)^n}{2^{n-1}} \int_{x_i \geq 0} dx_1 \cdots dx_n V(x_1) \cdots V(x_n) e^{-\alpha x_1 - \alpha x_2} \\ &\quad \left[e^{-\alpha|x_2-x_3|} - e^{-\alpha x_2} e^{-\alpha x_3} \right] \cdots \left[e^{-\alpha|x_n-x_1|} - e^{-\alpha x_n} e^{-\alpha x_1} \right] \end{aligned} \quad (7.6)$$

The most singular contribution comes from those permutations of $\{x_k\}$ with a specific maximal ordering with all large terms $|x_i - x_j|$.¹⁸

$$\text{Tr}(P_\alpha Q_\alpha^{n-1})_{\text{LO}} = \frac{(-1)^n}{2^{n-1}} \times \frac{2^{n-1}}{n!} \int_{0 \leq x_1 \leq \cdots \leq x_n} V(x_1) \cdots V(x_n) e^{-2\alpha x_n} \prod_{k=1}^{n-1} (1 - e^{-2\alpha x_k}) \quad (7.7)$$

Evaluating the leading logarithmic term integrating first in $x_n \in [x_{n-1}, \infty]$ and so on, one obtains

$$\text{Tr}(P_\alpha Q_\alpha^{n-1}) = \frac{(-1)^n}{n!} [-(2a\alpha \log \alpha)^n], \quad (7.8)$$

and therefore (7.3) reads at leading logarithmic order

$$\alpha = -\frac{\bar{V}}{2} \lambda + \sum_{k=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n!} \frac{\lambda^n}{\alpha^{n-1}} (a\alpha \log \alpha)^n. \quad (7.9)$$

The series sums to $\alpha - \alpha^{1-2a\lambda}$. Solving for α and manipulating up to subleading logarithmic terms $\lambda^n \log^m \lambda$ with $n > m + 1$, we get

$$\alpha = \left(-\frac{\bar{V}}{2} \lambda \right)^{\frac{1}{1-2a\lambda}} \simeq -\frac{\bar{V}}{2} e^{\frac{\log \lambda}{1-2a\lambda}} \simeq -\frac{\bar{V}}{2} e^{(1+2a\lambda) \log \lambda} = -\frac{\bar{V}}{2} \lambda e^{2a\lambda \log \lambda}, \quad (7.10)$$

that is (7.1).

¹⁸Alternatively, one may replace at this point V by its leading asymptotic and reduce to the problem in section (2) where computations may be done more explicitly.

8 Conclusions

In summary, we have reconsidered the one-dimensional Schrödinger problem (1.1) for long-range potentials in the class (1.4) with the aim of deriving an improved asymptotic expansion of the binding energy of the (unique) bound state that is present at weak coupling. Non-analytic terms are present at all orders in the coupling and are of course of infrared origin. Our setup is the simplest possible where such infrared problems occur and it is not possible to apply the powerful machinery of relativistic quantum field theory. In particular, there are no simple cut-off procedures that capture them accurately fixing the correct long-distance scales. We computed the general third order asymptotic expansion by methods coming from boundary layer theory. In principle, similar techniques may be applied to more difficult problems involving different behaviours at infinity, like for instance $V(x) \sim \log(x)/x^2$ or even $\log(x)/x^3$, where the extension is non trivial. It would be quite interesting to explore similar problems in more than one dimension.

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A All order resummation of LO and NLO infrared logarithms for the potential V_{II}

The quantization condition (2.2) may be written

$$(1 - 2\nu) K_\nu(\alpha) - 2\alpha K_{\nu-1}(\alpha) = 0, \quad \nu = \frac{1}{2}\sqrt{1-\lambda}. \quad (\text{A.1})$$

We multiply it by α^ν and expand at $\alpha \rightarrow 0$. After some simple manipulation we obtain

$$\alpha = \left(\frac{4^\nu (2\nu - 1) \Gamma(\nu)}{(2\nu + 1) \Gamma(-\nu)} \right)^{\frac{1}{2\nu}} \left[1 + \frac{13 - 4\nu^2}{4(4\nu^4 - 5\nu^2 + 1)} \alpha^2 + \mathcal{O}(\alpha^4) \right]. \quad (\text{A.2})$$

For $\lambda \rightarrow 0$, we have $\nu \rightarrow 1/2$ and $\alpha = \mathcal{O}(\lambda)$. Thus, (A.2) may be consistently solved iteratively $\alpha = \alpha^{(0)} + \alpha^{(1)} + \dots$, i.e.

$$\alpha^{(0)} = \left(\frac{4^\nu (2\nu - 1) \Gamma(\nu)}{(2\nu + 1) \Gamma(-\nu)} \right)^{\frac{1}{2\nu}}, \quad \alpha^{(1)} = \frac{13 - 4\nu^2}{4(4\nu^4 - 5\nu^2 + 1)} \left(\frac{4^\nu (2\nu - 1) \Gamma(\nu)}{(2\nu + 1) \Gamma(-\nu)} \right)^{\frac{1}{\nu}}. \quad (\text{A.3})$$

We can expand these expressions for $\lambda \rightarrow 0$ and introduce the quantity $L = \log \lambda + \gamma_E - \log 2 + \frac{1}{2}$. The terms in (A.3) of the form $\lambda^{n+1} L^n$ and $\lambda^{n+2} L^n$ are

$$\begin{aligned} \alpha^{(0)} &= \frac{1}{4} \lambda e^{\frac{1}{2} \lambda L} \left(1 - \frac{1}{4} \lambda + \frac{3}{8} \lambda^2 L \right) + \text{N}^2 \text{LO}, \\ \alpha^{(1)} &= \frac{1}{16} \lambda^2 e^{\frac{3}{2} \lambda L} + \text{N}^2 \text{LO}. \end{aligned} \quad (\text{A.4})$$

The third and higher corrections do not contribute. Summing the terms in (A.4) we prove the quoted result (2.4).

B Heuristic derivation of the leading infrared logarithm at third order

The following combination in (3.3)

$$(\text{tr} K_\alpha)^2 - \text{tr} K_\alpha^2, \quad (\text{B.1})$$

is essentially $F(\alpha)$ in (3.5). For this quantity we know that its small $\alpha \rightarrow 0^+$ expansion has leading term $\alpha (\# \log \alpha + \#)$.¹⁹ There are further correction $\mathcal{O}(\alpha^2)$, but an analysis of (3.5) shows that they do not contain $\log^2 \alpha$ enhancement factors. This means that the $\lambda^3 \log^2 \lambda$ term comes entirely from the very last term of (3.3), i.e.

$$\frac{\lambda^3}{3} \text{tr} K_\alpha^3. \quad (\text{B.2})$$

The general expression of the trace is

$$\begin{aligned} \text{tr} K_\alpha^3 &= \frac{1}{8\alpha^3} \int_{\mathbb{R}} dx dy dz V(x) V(y) V(z) e^{-\alpha(|x-y|+|x-z|+|y-z|)} \\ &= \frac{1}{8} \left(\frac{2}{\pi}\right)^{3/2} \int_{\mathbb{R}} d\omega_1 d\omega_2 d\omega_3 \frac{\tilde{V}(\omega_1 - \omega_2) \tilde{V}(\omega_1 - \omega_3) \tilde{V}(\omega_2 - \omega_3)}{(\alpha^2 + \omega_1^2)(\alpha^2 + \omega_2^2)(\alpha^2 + \omega_3^2)}. \end{aligned} \quad (\text{B.3})$$

The function $\tilde{V}(\omega)$ is in general a function of $|\omega|$. It is thus convenient to exploit the symmetry of the integrand of (B.3) with respect to permutations and write

$$\text{tr} K_\alpha^3 = \frac{3}{4} \left(\frac{2}{\pi}\right)^{3/2} \int_{\omega_1 < \omega_2 < \omega_3} d\omega_1 d\omega_2 d\omega_3 \frac{\tilde{V}(\omega_1 - \omega_2) \tilde{V}(\omega_1 - \omega_3) \tilde{V}(\omega_2 - \omega_3)}{(\alpha^2 + \omega_1^2)(\alpha^2 + \omega_2^2)(\alpha^2 + \omega_3^2)}. \quad (\text{B.4})$$

Finally, for $\omega > 0$, we write $\tilde{V}(\omega) = \tilde{V}(0) + \omega \tilde{V}'(0^+) + \dots$ and keep the term proportional to $\tilde{V}(0) (\tilde{V}'(0^+))^2$. We heuristically claim that this is the desired contribution containing the $\log^2 \alpha$ term. Using again $\tilde{V}(0) = \frac{1}{\sqrt{2\pi}} \bar{V}$ as well as (3.16), we reduce the calculation to

$$\text{tr} K_\alpha^3|_* = \frac{3 a^2 \bar{V}}{4\pi} \int_{\omega_1 < \omega_2 < \omega_3} d\omega_1 d\omega_2 d\omega_3 \frac{\omega_1^2 + \omega_1 \omega_2 - \omega_2^2 - 3\omega_1 \omega_3 + \omega_2 \omega_3 + \omega_3^2}{(\alpha^2 + \omega_1^2)(\alpha^2 + \omega_2^2)(\alpha^2 + \omega_3^2)}. \quad (\text{B.5})$$

where we denoted by $(\dots)|_*$ the fact that we are picking up a particular piece of $\text{tr} K_\alpha^3$. We now split

$$\text{tr} K_\alpha^3|_* = \frac{3 a^2 \bar{V}}{4\pi} \sum_{n=1}^6 I_n, \quad (\text{B.6})$$

where the 6 integrals I_n are associated to the 6 terms in the numerator of the r.h.s. of (B.5). All of them may be computed exactly after introducing a constraint $|\omega_i| < 1$ that gives convergence at large ω without changing the singular parts we are looking for. With some

¹⁹Here and in the following we shall denote by $\#$ numerical coefficients.

effort the partial results are:

$$\begin{aligned}
I_1 &= \frac{\pi^2}{2\alpha^2} + \frac{\pi \log \alpha - \frac{\pi^3}{6} - 2\pi + \pi \log 2}{\alpha} + \pi^2 + \mathcal{O}(\alpha), \\
I_2 &= \frac{12\pi \log^2 \alpha + 24\pi \log 2 \log \alpha + \pi^3 + 12\pi \log^2 2}{12\alpha} - 4 + \mathcal{O}(\alpha), \\
I_3 &= \frac{2\pi \log \alpha + \frac{\pi^3}{6} + 2\pi \log 2}{\alpha} + (-4 - \pi^2) + \mathcal{O}(\alpha), \\
I_4 &= \frac{12\pi \log^2 \alpha + 24\pi \log 2 \log \alpha + \pi^3 + 12\pi \log^2 2}{4\alpha} - 12 + \mathcal{O}(\alpha), \\
I_5 &= -2 + \mathcal{O}(\alpha), \\
I_6 &= \frac{\pi^2}{2\alpha^2} + \frac{\pi \log \alpha - \frac{\pi^3}{6} - 2\pi + \pi \log 2}{\alpha} + \pi^2 + \mathcal{O}(\alpha).
\end{aligned} \tag{B.7}$$

Summing up,

$$\begin{aligned}
\sum_{n=1}^6 I_n &= \frac{\pi^2}{\alpha^2} + \frac{\pi (24 \log^2 \alpha + 24(1 + 2 \log 2) \log(\alpha) + \pi^2 + 24(-1 + \log^2 2 + \log 2))}{6\alpha} \\
&\quad + (\pi^2 - 22) + \mathcal{O}(\alpha).
\end{aligned} \tag{B.8}$$

The term $\sim \alpha^{-2}$ is going to cancel against other terms, see (3.3), while the relevant piece in (B.2) turns out to be

$$\frac{\lambda^3}{3} \text{tr} K_\alpha^3 = a^2 \bar{V} \frac{\lambda^3}{\alpha} \log^2 \alpha + \dots, \tag{B.9}$$

where dots stand for terms we are not interested in. We conclude that the leading logarithmic part of $\alpha(\lambda)$ is obtained by inverting the expansion, (see (3.3), (3.11), etc.)

$$0 = 1 + \frac{\lambda}{2\alpha} \bar{V} + \lambda^2 a \bar{V} \log \alpha + \lambda^3 a^2 \bar{V} \log^2 \alpha + \dots, \tag{B.10}$$

that gives

$$\alpha = -\frac{\lambda}{2} \bar{V} \left(1 + 2a\lambda \log \lambda + 2a^2 \lambda \log^2 \lambda + \dots \right), \tag{B.11}$$

or, going back to (3.1),

$$c_{3,2} = -a^2 \bar{V}, \tag{B.12}$$

in agreement with table 1.

C Resummation of leading logarithms in a cut-off version of $V_{\text{II}}(x)$

To support the universal exponentiation of the leading logarithms as well as to show how it cannot be trivially extended to next-to-leading logarithms, we consider the illustrative problem associated with the following potential

$$V(x) = \begin{cases} -\frac{1}{4(1+x_0)^2} & \text{if } |x| \leq x_0, \\ -\frac{1}{4(1+|x|)^2} & \text{if } |x| > x_0. \end{cases} \quad x_0 > 0, \tag{C.1}$$

This is a simple modification of $V_{\text{II}}(x)$ that replaces it by the constant $V_{\text{II}}(x_0)$ for $|x| \leq x_0$. Thus, the asymptotic behaviour at infinity is unchanged. The study of this example is useful to see how the global shape of V affects the asymptotic expansion of its ground state energy. The exact ground state wave function is (with irrelevant arbitrary normalization)

$$\psi_0(x) = \begin{cases} \cos\left(x \frac{\sqrt{\lambda-4\alpha^2(x_0+1)^2}}{2(x_0+1)}\right) & \text{if } |x| \leq x_0, \quad x_0 > 0, \\ C \sqrt{|x|+1} K_{\frac{1}{2}\sqrt{1-\lambda}}(\alpha(|x|+1)) & \text{if } |x| > x_0. \end{cases} \quad (\text{C.2})$$

where C is fixed by the requirement of continuity in $|x| = x_0$. Quantization amounts to the requirement

$$\psi'_0(x_0^-) = \psi'_0(x_0^+), \quad (\text{C.3})$$

and gives a transcendental relation between α and λ . Expanding at small λ , we find

$$\begin{aligned} \sqrt{-E_0(\lambda)} = & \frac{2x_0+1}{4(x_0+1)^2} \lambda \left[1 + \frac{1}{2} L \lambda + \left(\frac{1}{8} L^2 + \frac{3x_0^2+10x_0+5}{8(x_0+1)^2} L \right. \right. \\ & \left. \left. - \frac{896x_0^6+5988x_0^5+15735x_0^4+19620x_0^3+12510x_0^2+3960x_0+495}{1440(x_0+1)^4(2x_0+1)^2} \right) \lambda^2 \right. \\ & + \left(\frac{1}{48} L^3 + \frac{3x_0^2+14x_0+7}{16(x_0+1)^2} L^2 \right. \\ & \left. \left. + \frac{1744x_0^6+12492x_0^5+31725x_0^4+38340x_0^3+24030x_0^2+7560x_0+945}{2880(x_0+1)^4(2x_0+1)^2} L + \text{const} \right) \lambda^4 + \dots \right], \end{aligned} \quad (\text{C.4})$$

$$L = \log \lambda + \gamma_E + \log \left(\frac{2x_0+1}{2(x_0+1)} \right) - \frac{8x_0^3+3x_0^2-6x_0-3}{6(x_0+1)^2(2x_0+1)}. \quad (\text{C.5})$$

Of course, the expansion (C.4) reduces to (2.3) for $x_0 = 0$ and is consistent with (4.39) and (4.40) for generic x_0 — the prefactor in (C.4) is $-\frac{1}{2}\bar{V}$. The leading logarithms are precisely the same as in (2.3). We pushed further the expansion (C.4) identifying the rational coefficients of the higher leading logarithms and they continue to agree with (2.3). This strongly supports the exponentiation (7.1), where the prefactor depends on the global shape of V , while the exponential is universal in the sense that it depends only on the coefficient of the leading term of V at infinity. Instead, the subleading logarithms depend on x_0 both through the scale entering the definition of L as well as in the remaining coefficient of the $\lambda^{n+2} L^n$ contributions.

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